

ON A PROPERTY OF SPECIAL TRIANGLE POINTS IN A SPECIFIC CONFIGURATION

YIMING XIANG

On May 20, 2017, Lin Tianqi posted the following problem on the WeChat public account “We Love Geometry”:

Problem 0.1. As shown in Figure 1, in $\triangle ABC$, I is the incenter and O is the circumcenter; $E = BI \cap AC$ and $F = CI \cap AB$. Points $U \in EF$ and $V \in BC$ are such that $I \in UV$ and $UV \perp OI$. Prove that $IV = 2UI$.

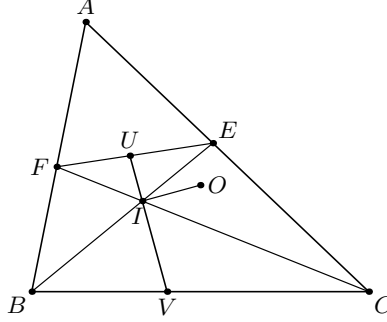


FIGURE 1. Problem 0.1.

On August 24, 2017, Jin Lei posted the following problem on the WeChat public account “We Love Geometry”:

Problem 0.2. As shown in Figure 2, in $\triangle ABC$, AH_a is an altitude, X_a is the midpoint of AH_a , $E = BX_a \cap AC$ and $F = CX_a \cap AB$. Points $U \in EF$ and $V \in BC$ are such that $X_a \in UV$ and $UV \perp OX_a$. Prove that $X_aV = 2UX_a$.

The author noticed that the structures of these two problems are very similar, and therefore began this investigation.

1. PRELIMINARIES

For convenience, unless otherwise stated, the letters below have the following meanings. In $\triangle ABC$, M_a, M_b, M_c are the midpoints of BC, CA, AB , and X_a, X_b, X_c are the midpoints of the altitudes AH_a, BH_b, CH_c . The points O, I, H, G, K are the circumcenter, incenter, orthocenter, centroid and symmedian point. The side lengths are a, b, c , the semiperimeter is p , the inradius is r and the circumradius is R . For a point P in the plane ($P \neq X$, since X_a, X_b, X_c are reserved as

Date: Revised April 14, 2020.

2020 Mathematics Subject Classification. Primary 51M04.

Key words and phrases. Trilinear polar, cevian triangle, isogonal conjugate, Thomson cubic.

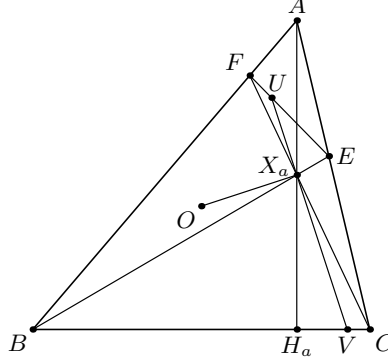
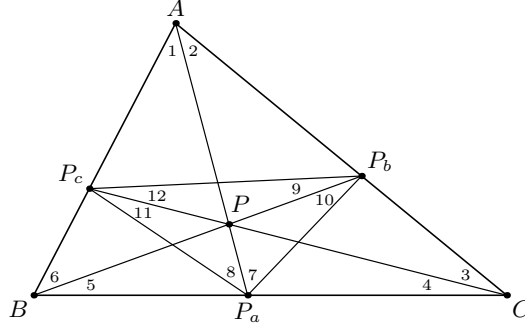


FIGURE 2. Problem 0.2.

above), $\triangle P_a P_b P_c$ denotes the cevian triangle of P with respect to $\triangle ABC$, where $P_a \in BC$, and so on, and P^* denotes the isogonal conjugate of P with respect to $\triangle ABC$. Finally, set

$$\frac{BP_a}{CP_a} = a_p, \quad \frac{CP_b}{AP_b} = b_p, \quad \frac{AP_c}{BP_c} = c_p.$$

Once the point P and $\triangle ABC$ are specified, we use $\angle 1$ to $\angle 12$ to denote the angles shown in Figure 3, and the symbol $[x]$ to denote $\cot \angle x$ ($x = 1, 2, \dots, 12$).

FIGURE 3. The angles $\angle 1, \dots, \angle 12$.

For ease of exposition, we introduce the following definition.

Definition 1.1. Let P be a point inside an arbitrary $\triangle ABC$, and let $U \in P_b P_c$ and $V \in BC$ be such that the line UPV is perpendicular to OP . If $PV = 2PU$, we say that P has *property Ω with respect to $A-BC$* . If P has property Ω with respect to each of $A-BC$, $B-CA$ and $C-AB$, we say that P *satisfies property Ω with respect to $\triangle ABC$* ; when the triangle and P are specified, we simply say that P satisfies property Ω . For the sake of rigor, we stipulate that the circumcenter satisfies property Ω , and that if $P_b P_c \parallel BC$ and $BC \perp OP$, then P has property Ω with respect to $A-BC$.

Note. When P lies outside the triangle there are completely analogous results, but the cases are too numerous to list one by one. In what follows, the author also considers only the case where P lies inside the triangle.

We now give some other definitions and lemmas needed in this paper.

Definition 1.2. The perspectrix of a triangle and the cevian triangle of a point with respect to it is called the *trilinear polar* of the point with respect to the triangle.

Lemma 1.3 (Cotangent rule). *In $\triangle ABC$, let X be a point on side AB with $XA : XB = m : n$, $\angle CXB = \alpha$ and $\angle ACX = \beta$. Then*

$$\begin{cases} (m+n) \cot \alpha = n \cot \angle A - m \cot \angle B, \\ m \cot \beta = (m+n) \cot \angle C + n \cot \angle A. \end{cases}$$

Proof. Drop the perpendicular $CD \perp AB$ with foot D , and let $CD = h$. Then

$$AX = h \cot \angle A - h \cot \alpha, \quad BX = h \cot \angle B + h \cot \alpha.$$

Since $m XB = n XA$ by hypothesis, this gives

$$(m+n) \cot \alpha = n \cot \angle A - m \cot \angle B.$$

Draw $XE \parallel BC$ meeting AC at E (Figure 4). By parallel lines, $AE : CE = m : n$. Noting that $\angle AEX = \angle C$, and applying the result just proved to the point E on side AC of $\triangle CAX$, we get

$$m \cot \beta = (m+n) \cot \angle C + n \cot \angle A.$$

This proves the lemma. □

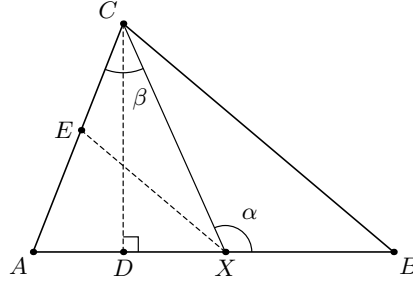


FIGURE 4. Theorem 1.3.

Lemma 1.4. *Let P be a point inside $\triangle ABC$. Then $\frac{BP_a}{CP_a} = \frac{S_{\triangle ABP}}{S_{\triangle ACP}}$, where $S_{\triangle XYZ}$ denotes the area of $\triangle XYZ$.*

This result is easy to prove, and we do not repeat it here.

Lemma 1.5. *In $\triangle ABC$, the tangents to $\odot O$ at A, B, C meet BC, CA, AB at P, Q, R respectively. Then P, Q, R are collinear (on the Lemoine line), and $PQR \perp OK$.*

Proof. As shown in Figure 5, by the basic properties of polars it suffices to prove that the polar of K with respect to $\odot O$ passes through P .

Extend AK to meet $\odot O$ again at F . Since AK is the A -symmedian of $\triangle ABC$, $ABFC$ is a harmonic quadrilateral. By the properties of harmonic quadrilaterals,

the tangents to $\odot O$ at A and F and the line BC are concurrent, so PF is tangent to $\odot O$ [1].

Note that AF is the polar of P with respect to $\odot O$, so the polar of P passes through K ; hence the polar of K with respect to $\odot O$ passes through P . This proves the lemma. $\square$

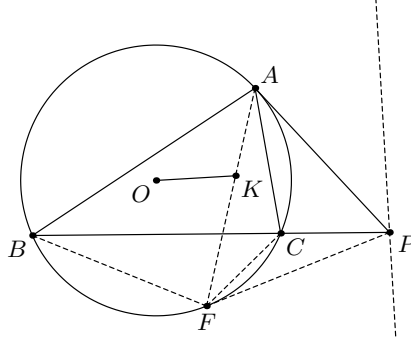


FIGURE 5. Theorem 1.5.

Lemma 1.6. *Let P be a point inside $\triangle ABC$, and let D, E, F be the feet of the perpendiculars from P to the three sides. Then*

$$CD = \frac{PE + PD \cdot \cos \angle C}{\sin \angle C}.$$

Proof. Suppose first that $\angle C$ is acute or right. Draw the line through P parallel to AC , meeting BC at X , and let Y be the foot of the perpendicular from X to AC (Figure 6). Then

$$\frac{PE}{\sin \angle C} = \frac{XY}{\sin \angle C} = CX, \quad PD \cdot \cot \angle C = PD \cdot \cot \angle PXD = DX.$$

Hence

$$CD = DX + XC = PD \cdot \cot \angle C + PE \cdot \csc \angle C = \frac{PE + PD \cdot \cos \angle C}{\sin \angle C}.$$

When $\angle C$ is obtuse,

$$\frac{PE}{\sin \angle C} = \frac{XY}{\sin \angle C} = CX, \quad PD \cdot \cot \angle C = -PD \cdot \cot \angle PXD = -DX,$$

hence

$$CD = -DX + XC = PD \cdot \cot \angle C + PE \cdot \csc \angle C = \frac{PE + PD \cdot \cos \angle C}{\sin \angle C}.$$

This proves the lemma. $\square$

Definition 1.7 (Trilinear coordinates). Fix a triangle ABC in the plane. The *trilinear coordinates* of a point in the plane of $\triangle ABC$ are an ordered triple, namely the ratio of its directed distances to the three sides. The sides are taken in the order BC, CA, AB , and each distance is measured positively along the outward normal of the corresponding side. The trilinear coordinates of a point P are written $(x : y : z)$; for example, the centroid has trilinear coordinates $(\frac{1}{a} : \frac{1}{b} : \frac{1}{c})$.

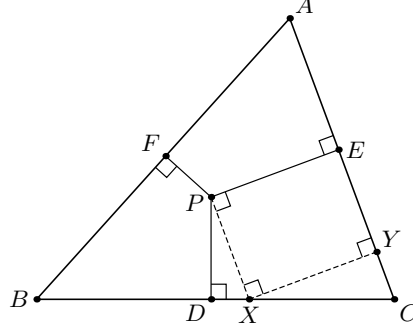


FIGURE 6. Theorem 1.6.

Lemma 1.8 ([2]). *Let P, Q, R be three distinct points with trilinear coordinates $(x_1 : y_1 : z_1)$, $(x_2 : y_2 : z_2)$, $(x_3 : y_3 : z_3)$ with respect to $\triangle ABC$. Then P, Q, R are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

2. INSPIRATION FROM THE TRILINEAR POLAR: AN EQUIVALENT CONDITION FOR PROPERTY Ω

The similarity between Problem 0.1 and Problem 0.2 led the author to think about the general case. Besides the incenter and the midpoint of an altitude, the centroid and the orthocenter also satisfy property Ω . While proving these problems, the author found that their proofs were all rather different, which is at odds with the similarity of the problems themselves. So the author began to wonder whether there might be a general method that works for all problems of this kind.

On May 21, 2017, TelvCohl, a plane geometry enthusiast from Chiayi, Taiwan, discovered a direct relationship between the problem and the trilinear polar. This explains, to some extent, the reason for the similarity, and turns the problem into one that is convenient to handle.

Let us first look at the theorem discovered by TelvCohl.

Theorem 2.1 ([3]). *In $\triangle ABC$, let t be the trilinear polar of the point P . The line through P parallel to t meets P_bP_c at U and BC at V . Then $PV = 2PU$.*

Proof. As shown in Figure 7, let P_bP_c meet BC at J , let P_aP_b meet AB at K , and let UV meet JA at L .

Note that CP_bAPBP_a is a complete quadrilateral, and in a complete quadrilateral each diagonal is divided harmonically by the other two. Hence B, A, P_c, K form a harmonic range, so JV, JL, JU, JK form a harmonic pencil. Since $LV \parallel t$, U is the midpoint of LV .

Similarly, JB, JU, JP, JA form a harmonic pencil, so V, U, P, L form a harmonic range, and by the properties of harmonic ranges, $LU \cdot PV = PU \cdot LV$. Substituting $LV = 2LU$ gives $PV = 2PU$. $\square$

Converse of Theorem 2.1. *In $\triangle ABC$, let t be the trilinear polar of the point P . A line through P meets P_bP_c at U and BC at V with $PV = 2PU$. Then $UV \parallel t$.*

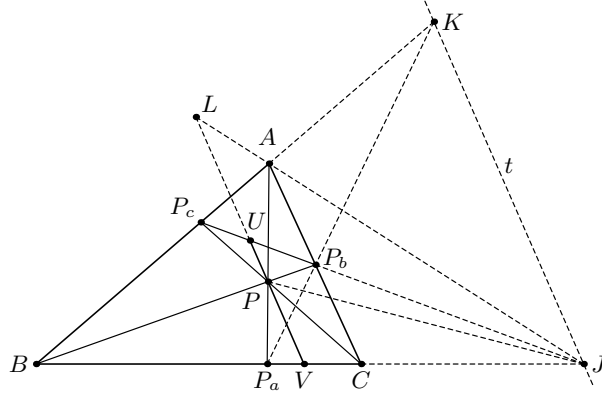


FIGURE 7. Theorem 2.1 and its converse.

Proof. As in Figure 7, V, U, P, L still form a harmonic range, so U is the midpoint of LV , and therefore $UV \parallel t$. $\square$

The segment-ratio statement that the problem asks us to prove is sometimes hard to handle, but by Theorem 2.1 it becomes a more concise and symmetric statement: it is turned into the problem of characterizing the trilinear polar. In many cases, Theorem 2.1 greatly simplifies the solution. We give three examples.

Problem 2.2. Prove that the incenter I has property Ω with respect to $A-BC$ (Problem 0.1).

Proof. Let I_1 be the excenter opposite A , and define I_2, I_3 similarly. Let O' be the circumcenter of $\triangle I_1 I_2 I_3$. Note that I and O are the orthocenter and the nine-point center of $\triangle I_1 I_2 I_3$, so I, O, O' are collinear.

Let $I_a I_c \cap AC = S$ and $I_b I_c \cap BC = X$ (Figure 8). Since C, A, I_b, S form a harmonic range and BI_b bisects $\angle ABC$, we have $BI_b \perp BS$; that is, S, I_3, I_1 are collinear. Similarly, X, I_3, I_2 are collinear.

Since A, C, I_3, I_1 are concyclic, S lies on the radical axis of $\odot(ABC)$ and $\odot(I_1 I_2 I_3)$; similarly for X . Hence SX is the radical axis of these two circles, so $SX \perp O'O$, i.e. $SX \perp OI$, and therefore $SX \parallel UV$. By Theorem 2.1, $IV = 2IU$. $\square$

A similar method shows that the excenters also satisfy property Ω .

Problem 2.3. Prove that the symmedian point K has property Ω with respect to $A-BC$.

Proof. As shown in Figure 9, let the tangents to $\odot O$ at A and B meet BC and AC at P and Q respectively. By Theorem 1.5, K_b, K_c, P are collinear, K_a, K_c, Q are collinear, and $OK \perp PQ$. So PQ is the trilinear polar of K with respect to $\triangle ABC$, and by Theorem 2.1, $KV = 2KU$. $\square$

Problem 2.4. Prove that X_b has property Ω with respect to $A-BC$.

Proof. We first prove that X_a has property Ω with respect to $A-BC$ (Problem 0.2). As shown in Figure 10, extend FE to meet BC at X , let Y be the midpoint of

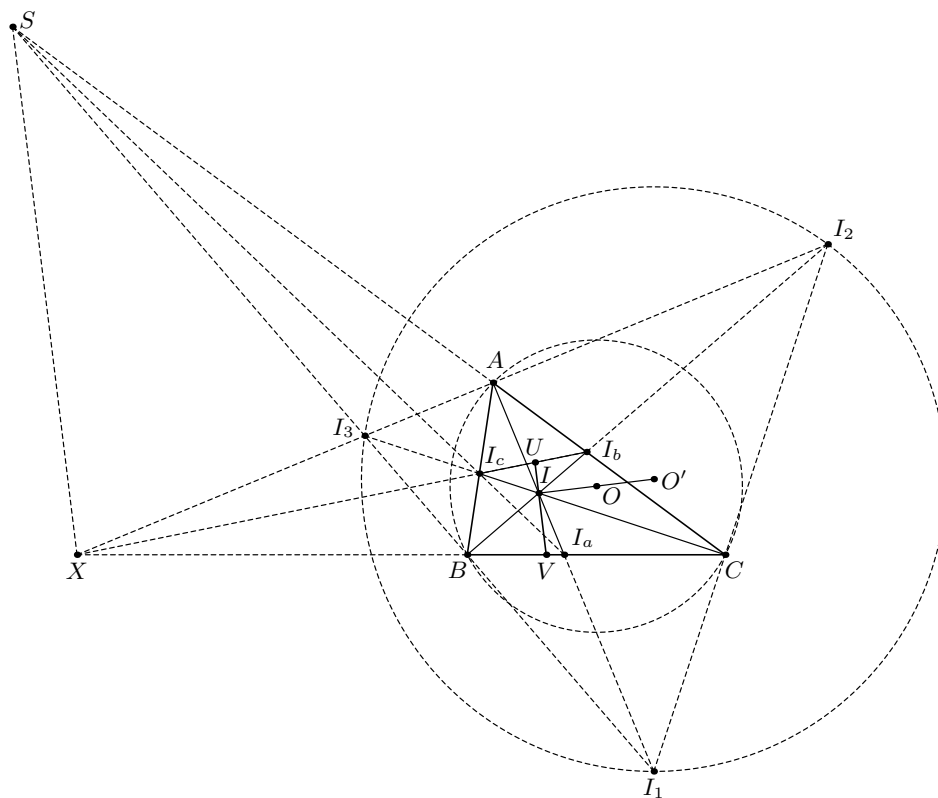


FIGURE 8. Problem 2.2.

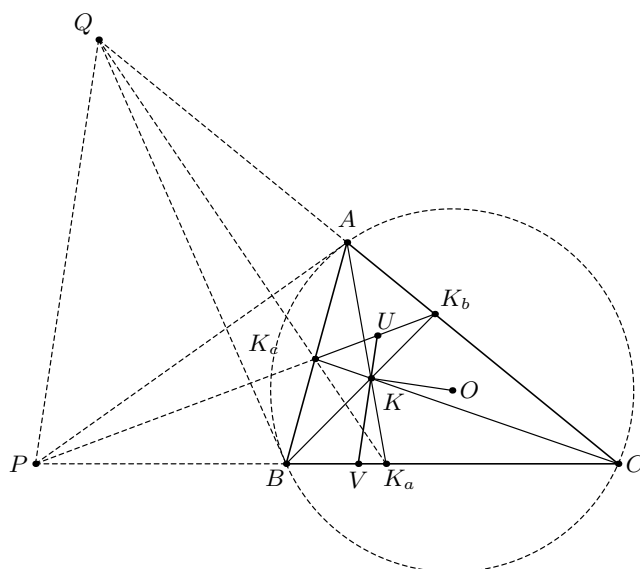


FIGURE 9. Problem 2.3.

$$YX_a^2 - X_aA^2 = YH_a^2 = YC \cdot YB = OY^2 - OA^2.$$

Applying Menelaus' theorem to $\triangle ZH_aX$ with the transversal UV ,

$$\frac{UZ \cdot X_a H_a \cdot VX}{UX \cdot X_a Z \cdot V H_a} = 1, \quad \text{so} \quad \frac{UZ}{UX} = \frac{1}{9}.$$

$$\frac{ZU \cdot X_a V \cdot H_a X}{ZX \cdot X_a U \cdot H_a V} = 1, \quad \text{so} \quad \frac{X_a V}{X_a U} = 2,$$

Returning to the original problem, as shown in [Figure 11](#), let $\triangle X_{b_a}X_{b_b}X_{b_c}$ be the cevian triangle of X_b with respect to $\triangle ABC$. Extend $X_{b_b}X_{b_c}$ to meet BC at L , extend $X_{b_a}X_{b_c}$ to meet AC at M , extend VU to meet AC at P , and let UV meet $X_{b_a}X_{b_c}$ at S .

If $X_{b_a}X_{b_c}$ is not parallel to AC , then the proof of Problem 0.2 above gives $PX_b = 2X_bS$. By the converse of Theorem 2.1, $ML \parallel UV$, and then by Theorem 2.1, $X_bV = 2X_bU$. $\square$

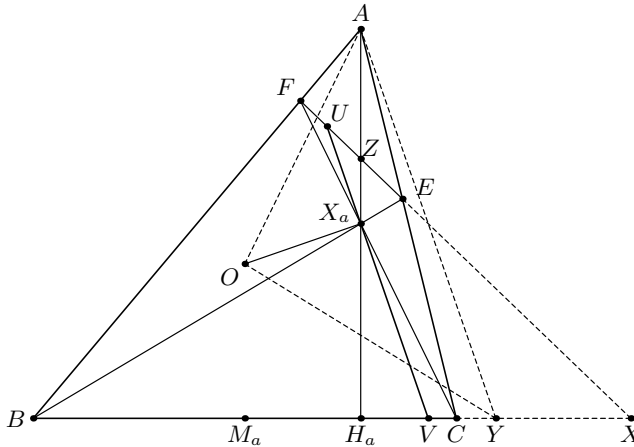


FIGURE 10. X_a has property Ω with respect to $A-BC$.

The statement of property Ω is symmetric only with respect to two of the vertices of the triangle. From [Theorem 2.1](#), however, we see that property Ω is symmetric in A , B and C ; the proof of [Problem 2.4](#) above is an instance of this symmetry. In general, we have the following theorem.

Theorem 2.5. *Let P be a point inside $\triangle ABC$. Then P has property Ω with respect to $A-BC \iff P$ satisfies property Ω .*

Therefore, from now on we no longer use the notation $A\text{--}BC$ and the like, and simply say that a point satisfies property Ω . We can now give the first equivalent condition for property Ω .

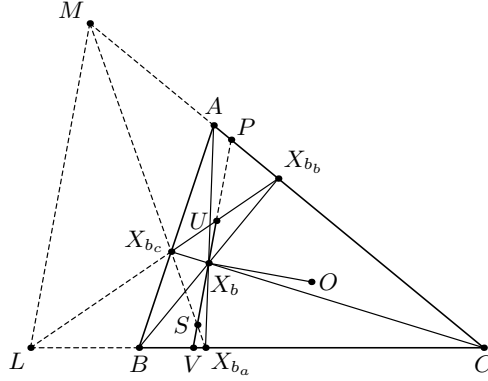


FIGURE 11. Problem 2.4.

Theorem 2.6. *Let P be a point inside $\triangle ABC$. Then P satisfies property Ω with respect to $\triangle ABC \iff$ the trilinear polar of P with respect to $\triangle ABC$ is parallel to the polar of P with respect to the circumcircle of $\triangle ABC$, or P is the circumcenter or the centroid of $\triangle ABC$.*

This follows directly from Theorem 2.1 and its converse; note that the circumcenter has no polar and the centroid has no trilinear polar.

Theorem 2.6 not only gives us an approach to problems of this kind, but also gives a very concise equivalent condition. In what follows we use it as the basis for transforming the problem further.

3. OTHER EQUIVALENT CONDITIONS FOR PROPERTY Ω

The author found that Theorem 2.1 is somewhat inadequate for some of the problems of this kind. For example, when proving that the isogonal conjugate of X_a satisfies property Ω , Theorem 2.1 seems to be of little help. So the author began to consider whether there are other ways to transform such problems.

Theorem 3.1. *Let X be a point inside an arbitrary $\triangle ABC$. Let a be the trilinear polar of X with respect to $\triangle ABC$, let b be the polar of X with respect to the circumcircle of $\triangle ABC$, and let c be the perspectrix of the cevian triangle and the circumcevian triangle of X with respect to $\triangle ABC$. Then $a \parallel b \iff a \parallel c$ and $a \parallel b \iff b \parallel c$.*

Proof. As shown in Figure 12, let $\triangle DEF$ be the cevian triangle and $\triangle GRJ$ the circumcevian triangle of X with respect to $\triangle ABC$. Let

$$\begin{aligned} DE \cap GR &= K, & AB \cap GR &= M, & AB \cap DE &= Q, \\ DF \cap GJ &= L, & JG \cap CA &= N, & CA \cap DF &= P. \end{aligned}$$

Applying Menelaus' theorem to $\triangle ADQ$ with the transversal GKM ,

$$\frac{DG \cdot AM \cdot QK}{DK \cdot QM \cdot AG} = 1, \quad \text{so} \quad \frac{AM}{QM} = \frac{DK \cdot AG}{DG \cdot QK}.$$

Similarly,

$$\frac{AN}{NP} = \frac{DL \cdot AG}{DG \cdot PL}.$$

Therefore

$$a \parallel b \iff \frac{AM \cdot NP}{QM \cdot AN} = 1 \iff \frac{DK \cdot PL}{QK \cdot DL} = 1 \iff a \parallel c.$$

Next, applying Menelaus' theorem to $\triangle AGM$ with the transversal DKQ ,

$$\frac{AD \cdot GK \cdot MQ}{AQ \cdot MK \cdot GD} = 1, \quad \text{so} \quad \frac{MQ}{AQ} = \frac{MK \cdot GD}{AD \cdot GK}.$$

Similarly,

$$\frac{NP}{AP} = \frac{NL \cdot GD}{AD \cdot GL}.$$

Therefore

$$a \parallel b \iff \frac{MQ \cdot AP}{AQ \cdot NP} = 1 \iff \frac{MK \cdot GL}{GK \cdot NL} = 1 \iff b \parallel c.$$

This proves the theorem. $\square$

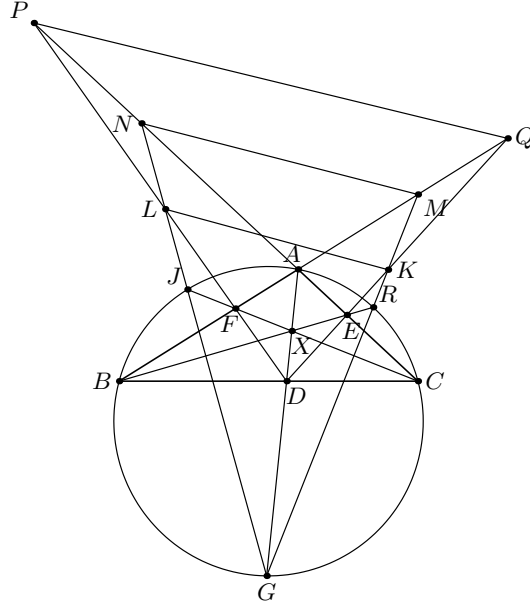


FIGURE 12. Theorem 3.1.

Theorem 3.2. *Let X be a point inside $\triangle ABC$. Then X satisfies property $\Omega \iff ([6] - [1])([3] - [8]) = ([3] - [2])([6] - [7])$.*

Proof. We continue to use Figure 12. Below, “the left-hand side” refers to the statement that X satisfies property Ω , and “the right-hand side” to the trigonometric identity. To make sure that the points defined above exist, we first deal with some special cases.

(1) $DF \parallel AC$ and $BA \neq BC$. If the left-hand side holds, then $AC = 2FD$, i.e. X is the centroid, and the right-hand side clearly holds. If the right-hand side holds, then since $[3] - [8] = [3] - [2] \neq 0$, it is equivalent to $[6] - [1] = [6] - [7]$; whether or not this quantity is 0, the left-hand side holds. So the conclusion holds.

(1') $DF \parallel AC$ and $BA = BC$. Then both sides hold, and the conclusion holds.

(2) $GJ \parallel AC$. Then X lies on the perpendicular bisector of segment AC . If the left-hand side holds, then $BA = BC$, so the right-hand side holds. If the right-hand side holds, then $([6] - [1])([3] - [8]) = 0$. If $[6] - [1] = 0$, then X is the circumcenter, and the left-hand side holds; if $[3] - [8] = 0$, then $BA = BC$, and the left-hand side holds. So the conclusion holds.

(3) $GJ \parallel FD$. As shown in Figure 13, F, D, C, A are concyclic, and $\angle BCX = \angle BAG = \angle BCG$. Let the line $UVXW$ be perpendicular to OX , with $U \in DF$, $V \in JG$, $W \in AC$. By the butterfly theorem, $XV = XW$. If the left-hand side holds, then $XV = 2XU$, so $DX = DG$ and $FX = FJ$. Hence $BC \perp XG$ and $AB \perp XJ$, so X is the orthocenter, and the right-hand side holds.

If the right-hand side holds, then $([3] - [2])([6] - [7]) = 0$. If $[3] = [2]$, then $BA = BC$, and the left-hand side holds. If $[6] = [7]$, then X is the orthocenter; as in Figure 13, $XV = XW$ and $UX = UV$, so $XW = 2XU$, and the left-hand side holds. So the conclusion holds.

General case. Suppose that no two lines are parallel within each of the three groups of lines DF, GJ, CA ; DE, GR, BA ; EF, JR, BC . As shown in Figure 12,

$$\frac{\sin(\angle 1 - \angle 6)}{\sin \angle 1} = \frac{AG}{MG}, \quad \frac{\sin(\angle 7 - \angle 6)}{\sin \angle 7} = \frac{DG}{KG}.$$

This gives

$$\frac{MG}{KG} = \frac{AG \cdot \sin \angle 1 \cdot \sin(\angle 7 - \angle 6)}{DG \cdot \sin \angle 7 \cdot \sin(\angle 1 - \angle 6)}, \quad \frac{NG}{LG} = \frac{AG \cdot \sin \angle 2 \cdot \sin(\angle 8 - \angle 3)}{DG \cdot \sin \angle 8 \cdot \sin(\angle 2 - \angle 3)}.$$

By Theorems 2.6 and 3.1, X satisfies property $\Omega \iff LK \parallel MN \iff \frac{MG}{KG} = \frac{NG}{LG}$. Substituting the above,

$$\begin{aligned} \frac{MG}{KG} = \frac{NG}{LG} &\iff \frac{\sin \angle 1 \cdot \sin(\angle 7 - \angle 6)}{\sin \angle 7 \cdot \sin(\angle 1 - \angle 6)} = \frac{\sin \angle 2 \cdot \sin(\angle 8 - \angle 3)}{\sin \angle 8 \cdot \sin(\angle 2 - \angle 3)} \\ &\iff \frac{\cos \angle 6 - \sin \angle 6 \cdot [1]}{\cos \angle 6 - \sin \angle 6 \cdot [7]} = \frac{\cos \angle 3 - \sin \angle 3 \cdot [2]}{\cos \angle 3 - \sin \angle 3 \cdot [8]} \\ &\iff \frac{[6] - [1]}{[6] - [7]} = \frac{[3] - [2]}{[3] - [8]} \\ &\iff ([6] - [1])([3] - [8]) = ([3] - [2])([6] - [7]). \end{aligned}$$

Hence the theorem holds in all cases. $\square$

Corollary 3.3. *Let X be a point inside $\triangle ABC$. Then X satisfies property Ω*

$$\begin{aligned} &\iff ([4] - [5])([1] - [10]) = ([4] - [9])([1] - [6]) \\ &\iff ([5] - [4])([2] - [11]) = ([5] - [12])([2] - [3]). \end{aligned}$$

Corollary 3.4. *Let P be a point inside $\triangle ABC$. Then P satisfies property Ω with respect to $\triangle ABC \iff ([1] - [7])([3] - [2]) = ([2] - [8])([6] - [1])$.*

Proof. By Theorem 3.2, P satisfies property Ω with respect to $\triangle ABC$

$$\begin{aligned} &\iff ([6] - [1])([3] - [8]) = ([3] - [2])([6] - [7]) \\ &\iff ([6] - [1])([3] - [2] + [2] - [8]) = ([3] - [2])([6] - [1] + [1] - [7]) \\ &\iff ([1] - [7])([3] - [2]) = ([2] - [8])([6] - [1]). \end{aligned} \quad \square$$

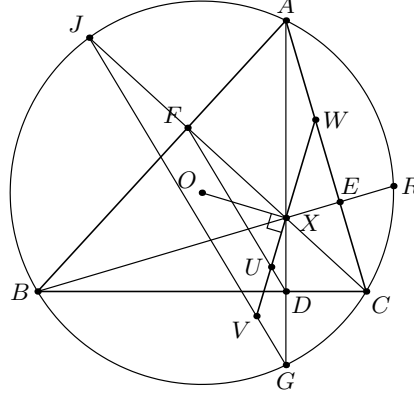


FIGURE 13. Case (3) in the proof of Theorem 3.2.

Corollary 3.5. *Let P be a point inside $\triangle ABC$. Then P satisfies property Ω with respect to $\triangle ABC$*

$$\begin{aligned} &\iff ([3] - [11])([5] - [4]) = ([2] - [3])([4] - [12]) \\ &\iff ([5] - [9])([1] - [6]) = ([6] - [10])([4] - [5]). \end{aligned}$$

After the transformation of Theorem 3.2, property Ω has become a trigonometric relation. However, it involves cotangents such as $[7]$ and $[8]$ that are hard to handle. Below we eliminate these cotangents through several theorems.

Theorem 3.6. $([1] - [7])(a_p - a_p b_p) = ([2] - [8])(1 - a_p b_p)$.

Proof. By Theorem 1.3, and since $a_p b_p c_p = 1$,

$$\begin{aligned} [1] &= \frac{a}{BP_a} \cdot \cot \angle A + \frac{CP_a}{BP_a} \cdot \cot \angle B = \frac{1 + a_p}{a_p} \cdot \cot \angle A + \frac{1}{a_p} \cdot \cot \angle B, \\ [2] &= \frac{a}{CP_a} \cdot \cot \angle A + \frac{BP_a}{CP_a} \cdot \cot \angle C = (1 + a_p) \cdot \cot \angle A + a_p \cdot \cot \angle C, \\ [7] &= \frac{b}{AP_b} \cdot \cot \angle AP_a C + \frac{CP_b}{AP_b} \cdot [2] = \frac{b_p + 1}{a_p + 1} \cdot (\cot \angle B - a_p \cdot \cot \angle C) + b_p \cdot [2] \\ &= b_p \cdot (1 + a_p) \cdot \cot \angle A + \frac{b_p + 1}{a_p + 1} \cdot \cot \angle B + \frac{a_p(a_p b_p - 1)}{a_p + 1} \cdot \cot \angle C, \\ [8] &= \frac{c}{AP_c} \cdot \cot \angle AP_a B + \frac{BP_c}{AP_c} \cdot [1] \\ &= \left(\frac{1}{c_p} + 1 \right) \cdot \left(\frac{1}{a_p + 1} \cdot (a_p \cdot \cot \angle C - \cot \angle B) \right) + \frac{1}{c_p} \cdot [1] \\ &= \frac{1 + a_p}{a_p c_p} \cdot \cot \angle A + \frac{1 - a_p c_p}{a_p c_p (a_p + 1)} \cdot \cot \angle B + \frac{a_p(c_p + 1)}{c_p(a_p + 1)} \cdot \cot \angle C \\ &= b_p(1 + a_p) \cdot \cot \angle A + \frac{b_p - 1}{a_p + 1} \cdot \cot \angle B + \frac{a_p b_p + 1}{b_p c_p + 1} \cdot \cot \angle C. \end{aligned}$$

Taking differences and simplifying,

$$\begin{aligned} [1] - [7] &= (1 - a_p b_p) \cdot \left(\frac{1 + a_p}{a_p} \cdot \cot \angle A + \frac{1}{a_p(1 + a_p)} \cdot \cot \angle B + \frac{a_p}{1 + a_p} \cdot \cot \angle C \right), \\ [2] - [8] &= (1 - b_p) \cdot \left((1 + a_p) \cdot \cot \angle A + \frac{1}{1 + a_p} \cdot \cot \angle B + \frac{a_p^2}{1 + a_p} \cdot \cot \angle C \right). \end{aligned}$$

Hence $([1] - [7])(a_p - a_p b_p) = ([2] - [8])(1 - a_p b_p)$. $\square$

Corollary 3.7. $([3] - [11])(a_p - 1) = ([4] - [12])(a_p - a_p b_p)$ and $([5] - [9])(a_p b_p - 1) = ([6] - [10])(a_p - 1)$.

Corollary 3.8. $\frac{[1] - [7]}{[2] - [8]}$ is an affine invariant.

Theorem 3.9. Let P be a point inside $\triangle ABC$. Then P satisfying property Ω is a necessary and sufficient condition for $[1] + [3] + [5] = [2] + [4] + [6]$.

Proof. We first prove the following lemma.

Lemma 3.10. Let P be a point inside $\triangle ABC$, and let d_a, d_b, d_c be the distances from P to BC, CA, AB . Then

$$([4]^2 - [5]^2)d_a^2 + ([2]^2 - [3]^2)d_b^2 + ([6]^2 - [1]^2)d_c^2 = 0.$$

Proof of Theorem 3.10. Let D_a, D_b, D_c be the feet of the perpendiculars from P to BC, CA, AB . Then

$$\begin{aligned} &([4]^2 - [5]^2)d_a^2 + ([2]^2 - [3]^2)d_b^2 + ([6]^2 - [1]^2)d_c^2 = 0 \\ \iff &\left(\frac{D_a C^2}{d_a^2} - \frac{D_a B^2}{d_a^2} \right) \cdot d_a^2 + \left(\frac{D_b A^2}{d_b^2} - \frac{D_b C^2}{d_b^2} \right) \cdot d_b^2 + \left(\frac{D_c B^2}{d_c^2} - \frac{D_c A^2}{d_c^2} \right) \cdot d_c^2 = 0 \\ \iff &PC^2 - PB^2 + PA^2 - PC^2 + PB^2 - PA^2 = 0, \end{aligned}$$

which proves the lemma. $\square$

Returning to the theorem, let $x = S_{\triangle BCP}$, $y = S_{\triangle ACP}$, $z = S_{\triangle ABP}$. By Theorem 1.4, $a_p = \frac{z}{y}$, $b_p = \frac{x}{z}$, $c_p = \frac{y}{x}$.

We first prove sufficiency. By Theorems 3.4 and 3.5, Theorem 3.6 and Theorem 3.7,

$$([6] - [1])(a_p - a_p b_p) = ([2] - [3])(a_p b_p - 1), \quad ([2] - [3])(1 - a_p) = ([4] - [5])(a_p b_p - a_p),$$

and therefore

$$\begin{aligned} ([6] - [1]) : [2] - [3] : [4] - [5] &= a_p b_p - 1 : a_p - a_p b_p : 1 - a_p \\ &= \frac{x}{y} - 1 : \frac{z}{y} - \frac{x}{y} : 1 - \frac{z}{y} \\ (1) \quad &= x - y : z - x : y - z. \end{aligned}$$

Since $x - y + z - x + y - z = 0$, we get $[6] - [1] + [2] - [3] + [4] - [5] = 0$, which proves sufficiency.

Next we prove necessity. By Theorem 3.10,

$$(2) \quad ([2]^2 - [3]^2)d_b^2 + ([6]^2 - [1]^2)d_c^2 = ([5]^2 - [4]^2)d_a^2 = d_a^2([5] - [4])([5] + [4]).$$

Using $[5] - [4] = [2] - [3] + [6] - [1]$, the right-hand side of (2) equals

$$([5] + [4])([2] - [3] + [6] - [1])d_a^2 = ([2] - [3])([5] + [4])d_a^2 + ([6] - [1])([5] + [4])d_a^2.$$

Rearranging,

$$([2] - [3])([5] + [4])d_a^2 - ([2]^2 - [3]^2)d_b^2 = ([6]^2 - [1]^2)d_c^2 - ([6] - [1])([5] + [4])d_a^2,$$

which is equivalent to

$$(3) \quad ([2] - [3])(([5] + [4])d_a^2 - ([2] + [3])d_b^2) = ([6] - [1])(([6] + [1])d_c^2 - ([5] + [4])d_a^2).$$

Note that $z = \frac{1}{2}c \cdot d_c = \frac{1}{2}([1] + [6])d_c^2$, and similarly $y = \frac{1}{2}([2] + [3])d_b^2$ and $x = \frac{1}{2}([4] + [5])d_a^2$. Substituting into (3) and simplifying gives $(x - y)([2] - [3]) = (z - x)([6] - [1])$. By the symmetry in x, y, z , (1) holds, and by the equivalences in [Theorems 3.4](#) and [3.5](#), necessity follows.

This completes the proof. $\square$

The proof of this theorem is rather tedious. If a reader can prove the conclusion directly by other means, without going through [Theorem 3.2](#), the proof would be greatly simplified.

Corollary 3.11. *Let P be a point inside $\triangle ABC$. Then P satisfies property $\Omega \iff P^*$ satisfies property Ω .*

Corollary 3.12. *Let P be a point inside $\triangle ABC$. Then P satisfies property Ω with respect to $\triangle ABC \iff \cot \angle PM_a C + \cot \angle PM_b A + \cot \angle PM_c B = 0$.*

The proof of this corollary needs only [Theorem 3.9](#) and [Theorem 1.3](#).

Theorem 3.13. *Let P be a point inside $\triangle ABC$. Then $[1] + [3] + [5] = [2] + [4] + [6] \iff P, G, P^*$ are collinear.*

Proof. Let P have trilinear coordinates $(x : y : z)$; then $P^* = (yz : xz : xy)$, and $G = (bc : ac : ab)$. By [Theorem 1.8](#), the right-hand side is equivalent to

$$\begin{vmatrix} bc & ac & ab \\ yz & xz & xy \\ x & y & z \end{vmatrix} = 0,$$

that is, $bcx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) = 0$.

Let D, E, F be the feet of the perpendiculars from P to the three sides. By [Theorem 1.6](#),

$$[2] = \frac{AE}{PE} = \frac{\frac{z+y \cos \angle A}{\sin \angle A}}{y} = \frac{z}{y \sin \angle A} + \cot \angle A = \frac{2zR}{ya} + \cot \angle A,$$

and so on. Hence the left-hand side is equivalent to

$$\begin{aligned} & [4] - [3] + [2] - [1] + [6] - [5] = 0 \\ \iff & \frac{y}{xc} - \frac{x}{yc} + \frac{z}{ya} - \frac{y}{za} + \frac{x}{zb} - \frac{z}{xb} = 0 \\ \iff & bcx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) = 0, \end{aligned}$$

which proves the theorem. $\square$

Theorem 3.14. *Let P be a point inside $\triangle ABC$. Then $[1] + [3] + [5] = [2] + [4] + [6] \iff [1]^2 + [3]^2 + [5]^2 = [2]^2 + [4]^2 + [6]^2$.*

Proof. It suffices to prove that for any $\triangle ABC$ and any point P inside it,

$$(1) \quad [1][3] + [3][5] + [5][1] = [2][4] + [4][6] + [6][2].$$

By the trigonometric identity

$$\cot x \cot y + \cot y \cot z + \cot z \cot x = \frac{\sin(x+y+z)}{\sin x \sin y \sin z} + 1,$$

(1) is equivalent to $\frac{\sin \angle 2 \cdot \sin \angle 4 \cdot \sin \angle 6}{\sin \angle 1 \cdot \sin \angle 3 \cdot \sin \angle 5} = 1$, which is exactly the trigonometric form of Ceva's theorem. $\square$

4. THE THOMSON CUBIC

So far we have found many points satisfying property Ω . Do they all lie on some curve?

By [Theorem 3.13](#), every point P inside the triangle that satisfies property Ω makes P, G, P^* collinear. In general, the locus of points such that the line through the point and its isogonal conjugate passes through a fixed point (called the *pivot*) is called a *pivotal isocubic*. A computation using [Theorem 1.8](#) shows that a pivotal isocubic has the general equation

$$f_1 x(y^2 - z^2) + f_2 y(z^2 - x^2) + f_3 z(x^2 - y^2) = 0,$$

where the pivot F has trilinear coordinates $(f_1 : f_2 : f_3)$.

It is not hard to see that the pivotal isocubic with the centroid as pivot has the equation (which is also an equivalent condition for property Ω)

$$bcx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) = 0.$$

In fact, it already appeared in the proof of [Theorem 3.13](#). Now it is clear that the points satisfying property Ω do lie on a curve. The pivotal isocubic with the centroid as pivot is called the *Thomson cubic* ([Figure 14](#)). Because of its rich geometric properties, many results about this curve can be found in *Cubics in the Triangle Plane* [\[4\]](#).

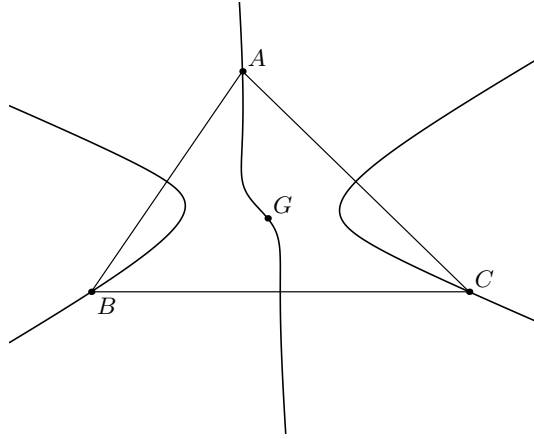


FIGURE 14. The Thomson cubic.

5. SUMMARY AND OUTLOOK

We summarize the equivalent conditions for property Ω obtained above. Let P be a point inside $\triangle ABC$, and let $UV \perp OP$ with $P \in UV$, $U \in P_bP_c$, $V \in BC$. Let a be the trilinear polar of P with respect to $\triangle ABC$, let b be the polar of P with respect to the circumcircle of $\triangle ABC$, and let c be the perspectrix of the cevian triangle of P and its circumcevian triangle with respect to $\triangle ABC$. Let P^* be the isogonal conjugate of P with respect to $\triangle ABC$, and let P have trilinear coordinates $(x : y : z)$. The following is a list of equivalent conditions for property Ω :

- (1) $UV \parallel a$;
- (2) $a \parallel b$;
- (3) $a \parallel c$;
- (4) $b \parallel c$;
- (5) $([6] - [1])([3] - [8]) = ([3] - [2])([6] - [7])$, and the analogous relations;
- (6) $([1] - [7])([3] - [2]) = ([2] - [8])([6] - [1])$, and the analogous relations;
- (7) $[1] + [3] + [5] = [2] + [4] + [6]$;
- (8) $\cot \angle PM_aC + \cot \angle PM_bA + \cot \angle PM_cB = 0$;
- (9) P, G, P^* are collinear;
- (10) $[1]^2 + [3]^2 + [5]^2 = [2]^2 + [4]^2 + [6]^2$;
- (11) $bzx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) = 0$.

The author proposes here a related conjecture, and hopes that readers can resolve it.

Conjecture 5.1. *Let P be a point inside $\triangle ABC$, and let P^* be its isogonal conjugate with respect to $\triangle ABC$. The line through P perpendicular to PM_a meets BC at T_a ; define T_b, T_c similarly. Then P, G, P^* are collinear $\iff T_a, T_b, T_c$ are collinear; moreover, in this case $T_aT_bT_c \perp PP^*$ (Figure 15).*

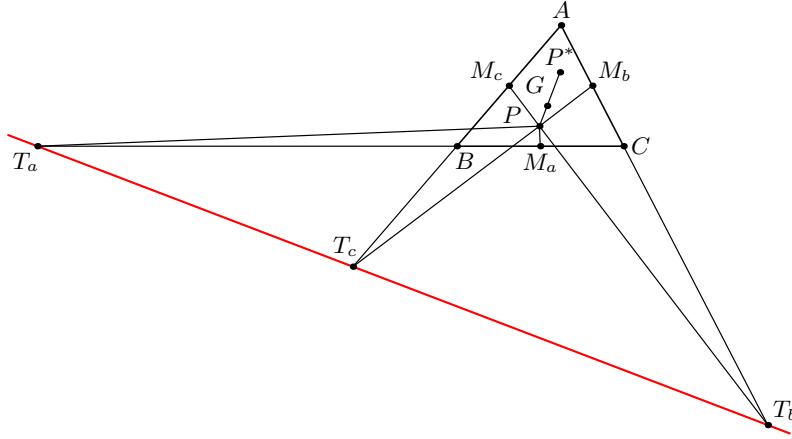


FIGURE 15. The conjecture.

6. CONCLUDING REMARKS

The author hopes that this paper will serve as a modest first step that attracts the attention of geometry enthusiasts, and that answers to the questions left open

here will be found. Special thanks go to my teacher Lu Sheng for guidance. In writing this paper, the author not only enjoyed exploring the problems, but also gained some training in expressing mathematics rigorously.

REFERENCES

1. Cao Jueyun and Ye Zhonghao, *Harmonic quadrilaterals and their applications*, Middle School Mathematics (Zhongdeng Shuxue) (2016), no. 1, 1–2 (Chinese).
2. Wu Yuechen, *Trilinear coordinates and triangle centers*, Harbin Institute of Technology Press, 2015, pp. 5–10 (Chinese).
3. TelvCohl, *A generalization of a problem*, Pure Geometry Forum (Baidu Tieba), <http://tieba.baidu.com/p/5125232674> (Chinese).
4. B. Gibert, *K002: the Thomson cubic*, Cubics in the Triangle Plane, <https://bernard-gibert.pagesperso-orange.fr/Exemples/k002.html>.

SHANGHAI WEIYU HIGH SCHOOL, SHANGHAI 200231, CHINA